MODEL ANSWER

Write your name here

Surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number	Candidate Number

Further Mathematics

Advanced
Paper 2: Core Pure Mathematics 2

Sample Assessment Material for first teaching September 2017

Time: 1 hour 30 minutes

Paper Reference **9FM0/02**

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 7 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

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Answer ALL questions. Write your answers in the spaces provided.

1. The roots of the equation

$$x^3 - 8x^2 + 28x - 32 = 0$$

are α, β and γ

Without solving the equation, find the value of

(i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

(ii) $(\alpha + 2)(\beta + 2)(\gamma + 2)$

(iii) $\alpha^2 + \beta^2 + \gamma^2$

(8)

i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$x^3 - 8x^2 + 28x - 32 = 0$

$a = 1$
$b = -8$
$c = 28$
$d = -32$

$\alpha + \beta + \gamma = \frac{-b}{a} = 8$ — 1

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = \frac{28}{1} = 28$ — 2

$\alpha\beta\gamma = \frac{-d}{a} = \frac{32}{1} = 32$ — 3

so $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{28}{32}$ ~ 2 ~ 3

$= \boxed{\frac{7}{8}}$

ii) $(\alpha+2)(\beta+2)(\gamma+2) = (\alpha\beta + 4 + 2\alpha + 2\beta)(\gamma + 2)$

$= (\alpha\beta\gamma + 4\gamma + 2\alpha\gamma + 2\beta\gamma + 2\alpha\beta + 8 + 4\alpha + 4\beta)$

$= \alpha\beta\gamma + 4(\alpha + \beta + \gamma) + 2(\alpha\gamma + \gamma\beta + \alpha\beta) + 8$

$= 32 + 4(8) + 2(28) + 8$

$= \boxed{128}$

iii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 8^2 - 2(28)$

$= \boxed{8}$

Question 1 continued

(Total for Question 1 is 8 marks)

2. The plane Π_1 has vector equation

$$\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 5$$

(a) Find the perpendicular distance from the point (6, 2, 12) to the plane Π_1

(3)

The plane Π_2 has vector equation

$$\mathbf{r} = \lambda(2\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} - 2\mathbf{k})$$

where λ and μ are scalar parameters.

(b) Show that the vector $-\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is perpendicular to Π_2

(2)

(c) Show that the acute angle between Π_1 and Π_2 is 52° to the nearest degree.

(3)

a) perpendicular distance $= \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}}$

$= \dfrac{3(6) - 4(2) + 2(12) + (-5)}{\sqrt{3^2 + 4^2 + 2^2}}$

$= \boxed{\sqrt{29}}$

b) $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ are both vectors lying on Π_2

$\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = -2 - 3 + 5 = 0$ //

$\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = -1 + 3 - 2 = 0$ //

so the given vector must be perpendicular to Π_2.

c) using normals:

$\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} = -3 + 12 + 2 = 11$

$\left|\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}\right| = \sqrt{3^2 + (-4)^2 + 2^2} = \sqrt{29}$

$\left|\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}\right| = \sqrt{(-1)^2 + (-3)^2 + 1^2} = \sqrt{11}$

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Question 2 continued

so $\cos\theta = \frac{11}{\sqrt{29 \times 11}}$

$= \frac{\sqrt{319}}{29}$

$\theta = \arccos \frac{\sqrt{319}}{29}$

$= \boxed{52°}$

(Total for Question 2 is 8 marks)

3. (i)

$$\mathbf{M} = \begin{pmatrix} 2 & a & 4 \\ 1 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

where a is a constant.

(a) For which values of a does the matrix $\mathbf{M}$ have an inverse?

(2)

Given that $\mathbf{M}$ is non-singular,

(b) find $\mathbf{M}^{-1}$ in terms of a

(4)

(ii) Prove by induction that for all positive integers n,

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$$

(6)

a) if det M=0 then M won't have an inverse

$$\det M = 2\begin{vmatrix} -1 & -1 \\ 2 & -1 \end{vmatrix} - a\begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} + 4\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 0$$

$$2(1+2) - a(-1-1) + 4(2-1) = 0$$

$$6 + 2a + 4 = 0$$

$$2a = -10$$

$$a = -5$$

so M will have an inverse when $\boxed{a \neq -5}$

b) matrix of minors: $\begin{pmatrix} \begin{vmatrix} -1 & -1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} \\ \begin{vmatrix} a & 4 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & a \\ -1 & 2 \end{vmatrix} \\ \begin{vmatrix} a & 4 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & a \\ 1 & -1 \end{vmatrix} \end{pmatrix}$

$$= \begin{pmatrix} 3 & -2 & 1 \\ -a-8 & 2 & a+4 \\ 4-a & -6 & -2-a \end{pmatrix}$$

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Question 3 continued

changing signs: $\begin{pmatrix} 3 & 2 & 1 \\ a+8 & 2 & -a-4 \\ 4-a & 6 & -2-a \end{pmatrix}$

transpose: $\begin{pmatrix} 3 & a+8 & 4-a \\ 2 & 2 & 6 \\ 1 & -a-4 & -2-a \end{pmatrix}$

$\therefore M^{-1} = \frac{1}{2a+10}\begin{pmatrix} 3 & a+8 & 4-a \\ 2 & 2 & 6 \\ 1 & -a-4 & -2-a \end{pmatrix}$

ii) $n=1$: $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^1$ = LHS = $\begin{pmatrix} 3 & 0 \\ 3(3-1) & 1 \end{pmatrix}$

$= \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ = RHS //

$\therefore$ true for $n=1$ //

assume true for $n=k$,

ie $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}$

consider $n=k+1$,

$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$

$= \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$

$= \begin{pmatrix} 3^{k+1} & 0 \\ 3^2(3^k-1)+6 & 1 \end{pmatrix}$

$3^2(\quad -1)+6 = 9(3^k)-9+6$

$= 9(3^k)-3$

$= 3(3\times 3^k - 1)$

$= 3(3^{k+1}-1)$

hence $\Rightarrow \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1}-1) & 1 \end{pmatrix}$ $\therefore$ true for $n=k+1$

so given relationship is true for $n=1$ and $n=n+1$ when assumed true for $n=k$

$\therefore$ by mathematical induction true for all $n \in \mathbb{Z}^+$

(Total for Question 3 is 12 marks)

4. A complex number z has modulus 1 and argument θ.

(a) Show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \qquad n \in \mathbb{Z}^+$$

(2)

(b) Hence, show that

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

(5)

a) $z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \frac{1}{z^n} = (\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

$\therefore z^n + \frac{1}{z^n} = \cos n\theta + \cos n\theta + i\sin n\theta - i\sin n\theta$

$= \boxed{2\cos n\theta}$

b) $\left(z + \frac{1}{z}\right)^4 = z^4 + 4(z)^3\left(\frac{1}{z}\right) + 6(z)^2\left(\frac{1}{z}\right)^2 + 4(z)\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4$

$= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$

$= 2\cos 4\theta + 4 \times 2\cos 2\theta + 6$

$= 2\cos 4\theta + 8\cos 2\theta + 6$

hence $(2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$

$16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$\cos^4\theta = \frac{1}{16}(2\cos 4\theta + 8\cos 2\theta + 6)$

$= \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

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Question 4 continued

(Total for Question 4 is 7 marks)

5.

$$y = \sin x \sinh x$$

(a) Show that $\dfrac{d^4y}{dx^4} = -4y$

(4)

(b) Hence find the first three non-zero terms of the Maclaurin series for y, giving each coefficient in its simplest form.

(4)

(c) Find an expression for the nth non-zero term of the Maclaurin series for y.

(2)

a) $y = \sin x \sinh x$

PRODUCT RULE

$y' = \cos x \sinh x + \sin x \cosh x$

$y'' = -\sin x \sinh x + \cos x \cosh x + \sin x \sinh x + \cos x \cosh x$

$= 2\cos x \cosh x$

$y''' = -2\sin x \cosh x + 2\cos x \sinh x$

$y'''' = -2\cos x \cosh x - 2\sin x \sinh x - 2\sin x \sinh x + 2\cos x \cosh x$

$= -4\sin x \sinh x$

$= -4y$

b) let $f(x) = y$,

then $f^n(x) = 0$ when $n \neq 2 + 4d$ where $d \geq 0$, $d \in \mathbb{Z}^+$

$f'(0) = 0$

$f^2(0) = 2$ //

$f^3(0) = 0$

$f^4(0) = 0$

$f^5(0) = 0$

$f^6(0) = -4y^{12} = -8$ //

only terms involving $\dfrac{d^2y}{dx^2}$ will be non-zero

$\boxed{f^n(0) = \left(\dfrac{d^ny}{dx^n}\right)_{x=0}}$

so for first 3 non-zero terms ...

$\left(\dfrac{d^2y}{dx^2}\right)_{x=0} = 2$

$\left(\dfrac{d^6y}{dx^6}\right)_{x=0} = -8$

$\left(\dfrac{d^{10}y}{dx^{10}}\right)_{x=0} = -4(-8) = 32$

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Question 5 continued

hence $y \simeq \frac{x^2}{2!}(2) + \frac{x^6}{6!}(-8) + \frac{x^{10}}{10!}(32)$

$$\boxed{y \simeq x^2 - \frac{x^6}{90} + \frac{x^{10}}{113400}}$$

c) looking at the powers from the answer to (b) and some earlier working,

2 . . . 6 . . . 10

the 'powers' form an arithmetic series with $a = 2$ and $d = 4$

so the n^{th} non-zero term will include an x term to the power of $(2 + (n-1) \times 4)$

$= 4n - 2$

and notice that $\left(\frac{d^n y}{dx^n}\right)_{x=0} = -4\left(\frac{d^{n-1} y}{dx^{n-1}}\right)_{x=0}$

the values of $\left(\frac{d^n y}{dx^n}\right)_{x=0}$ form a geometric series

2 . . .	-8 . . .	32
a	ar	ar^2
$2 \times (-4)^0$	$2 \times (-4)^1$	$2 \times (-4)^2$

$(n^{th} \text{ term} = ar^{n-1})$

this leads us to the n^{th} non-zero term being

$$-2(-4)^{n-1} \times \frac{x^{4n-2}}{(4n-2)!} \quad //$$

(Total for Question 5 is 10 marks)

6. (a) (i) Show on an Argand diagram the locus of points given by the values of z satisfying

$$|z - 4 - 3\mathbf{i}| = 5$$

Taking the initial line as the positive real axis with the pole at the origin and given that $\theta \in [\alpha, \alpha + \pi]$, where $\alpha = -\arctan\left(\frac{4}{3}\right)$,

(ii) show that this locus of points can be represented by the polar curve with equation

$$r = 8\cos\theta + 6\sin\theta$$

(6)

The set of points A is defined by

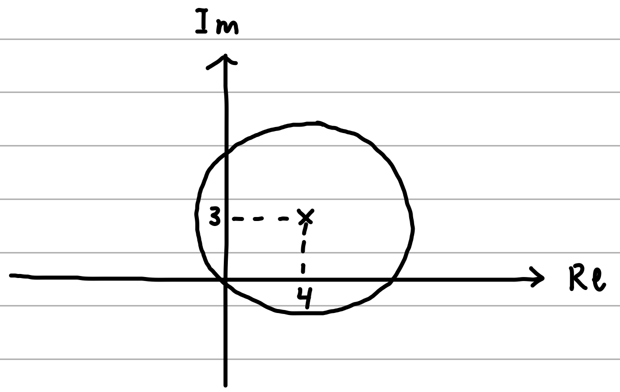
$$A = \left\{z : 0 \leqslant \arg z \leqslant \frac{\pi}{3}\right\} \cap \left\{z : |z - 4 - 3\mathbf{i}| \leqslant 5\right\}$$

(b) (i) Show, by shading on your Argand diagram, the set of points A.

(ii) Find the **exact** area of the region defined by A, giving your answer in simplest form.

(7)

a i) circle centre (4, 3), radius = 5, passes through O



ii) $|z - 4 - 3i| = 5$

$|x + iy - 4 - 3i| = 5$

$|(x-4) + (y-3)i| = 5$

$\therefore (x-4)^2 + (y-3)^2 = 25$

$x = r\cos\theta$

$y = r\sin\theta$

$\therefore (r\cos\theta - 4)^2 + (r\sin\theta - 3)^2 = 25$

$r^2\cos^2\theta - 8r\cos\theta + 16 + r^2\sin^2\theta - 6r\sin\theta + 9 = 25$

Question 6 continued

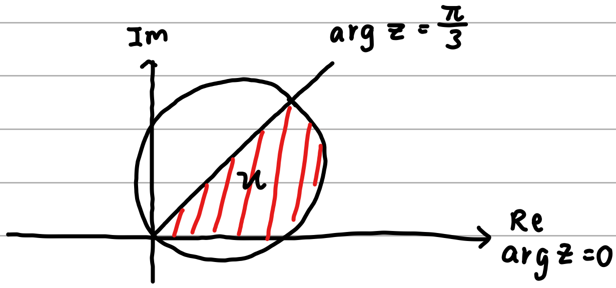
$r^2(\sin^2\theta + \cos^2\theta) - 8r\cos\theta - 6r\sin\theta = 25 - 25$

$\therefore\ r^2 = 8r\cos\theta + 6r\sin\theta$

$\div r$

$r = 8\cos\theta + 6\sin\theta$

bi)



region above real axis, below the half line ($\arg z = \frac{\pi}{3}$) and within the circle

ii) Area $= \frac{1}{2}\int_0^{\pi/3} r^2\, d\theta$

$= \frac{1}{2}\int_0^{\pi/3} (8\cos\theta + 6\sin\theta)^2\, d\theta$

$= \frac{1}{2}\int_0^{\pi/3} [64\cos^2\theta + 36\sin^2\theta + 96\sin\theta\cos\theta]\, d\theta$

$= \frac{1}{2}\int_0^{\pi/3} [36(\sin^2\theta + \cos^2\theta) + 28\cos^2\theta + 96\sin\theta\cos\theta]\, d\theta$

$\cos 2\theta = 2\cos^2\theta - 1$
$\rightarrow \cos^2\theta = \frac{\cos 2\theta + 1}{2}$

$= \frac{1}{2}\int_0^{\pi/3} \left[36 + 28\left(\frac{\cos 2\theta + 1}{2}\right) + 48\sin 2\theta\right] d\theta$

$= \frac{1}{2}\int_0^{\pi/3} [36 + 14\cos 2\theta + 14 + 48\sin 2\theta]\, d\theta$

$= \frac{1}{2}\int_0^{\pi/3} [50 + 14\cos 2\theta + 48\sin 2\theta]\, d\theta$

$= \frac{1}{2}\left[50\theta + 7\sin 2\theta - 24\cos 2\theta\right]_0^{\pi/3}$

$= \frac{1}{2}\left[\frac{50\pi}{3} + \frac{7\sqrt{3}}{2} + 12\right] - \frac{1}{2}[-24]$

$= \frac{1}{2}\left[\frac{50\pi}{3} + \frac{7\sqrt{3}}{2} + 36\right]$

$= \boxed{\frac{25\pi}{3} + \frac{7\sqrt{3}}{4} + 18}$

(Total for Question 6 is 13 marks)

7. At the start of the year 2000, a survey began of the number of foxes and rabbits on an island.

At time t years after the survey began, the number of foxes, f, and the number of rabbits, r, on the island are modelled by the differential equations

$$\frac{df}{dt} = 0.2f + 0.1r$$

$$\frac{dr}{dt} = -0.2f + 0.4r$$

(a) Show that $\frac{d^2 f}{dt^2} - 0.6\frac{df}{dt} + 0.1f = 0$

(3)

(b) Find a general solution for the number of foxes on the island at time t years.

(4)

(c) Hence find a general solution for the number of rabbits on the island at time t years.

(3)

At the start of the year 2000 there were 6 foxes and 20 rabbits on the island.

(d) (i) According to this model, in which year are the rabbits predicted to die out?

(ii) According to this model, how many foxes will be on the island when the rabbits die out?

(iii) Use your answers to parts (i) and (ii) to comment on the model.

(7)

a) $\frac{d^2f}{dt^2} - 0.6\frac{df}{dt} + 0.1f = 0$

notice there is no r in the above equation. so we want to essentially "eliminate" r

$\frac{df}{dt} = 0.2f + 0.1r$

×10

$10\frac{df}{dt} = 2f + r$

$\therefore r = 10\frac{df}{dt} - 2f$

$\therefore \frac{dr}{dt} = 10\frac{d^2f}{dt^2} - \frac{2df}{dt}$

and we are told $\frac{dr}{dt} = -0.2f + 0.4r$

$\therefore 10\frac{d^2f}{dt^2} - 2\frac{df}{dt} = -0.2f + 0.4r$

$10\frac{d^2f}{dt^2} - 2\frac{df}{dt} + 0.2f - 0.4r = 0$

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Question 7 continued

$$10\frac{d^2f}{dt^2} - 2\frac{df}{dt} + 0.2f - 0.4\left(10\frac{df}{dt} - 2f\right) = 0$$

$$10\frac{d^2f}{dt^2} - 6\frac{df}{dt} + f = 0$$

$\div 10$

$$\frac{d^2f}{dt^2} - 0.6\frac{df}{dt} + 0.1f = 0$$

b) AUX: $\lambda^2 - 0.6\lambda + 0.1 = 0$

$$\lambda = \frac{0.6 \pm \sqrt{0.6^2 - 4(0.1)}}{2} = 0.3 \pm 0.1i$$

so $f = e^{0.3t}(A\sin 0.1t + B\cos 0.1t)$

c) $\frac{df}{dt} = 0.3e^{0.3t}(A\sin 0.1t + B\cos 0.1t) + e^{0.3t}(0.1A\cos 0.1t - B\sin 0.1t \times 0.1)$

$= e^{0.3t}(0.3A\sin 0.1t + 0.1A\cos 0.1t + 0.3B\cos 0.1t - 0.1B\sin 0.1t)$

but $r = 10\frac{df}{dt} - 2f$ (from a)

$\therefore r = e^{0.3t}(3A\sin 0.1t + A\cos 0.1t + 3B\cos 0.1t - B\sin 0.1t) - 2f$

so $r = e^{0.3t}\left[(3A - B)\sin 0.1t + (A + 3B)\cos 0.1t\right] - 2e^{0.3t}(A\sin 0.1t + B\cos 0.1t)$

$r = e^{0.3t}\left[(A - B)\sin 0.1t + (A + B)\cos 0.1t\right]$

di) $t = 0$, $f = 6$: $6 = B$

$t = 0$, $r = 20$: $20 = A + B$

so $A = 14$

at $r = 0$ $\quad 0 = e^{0.3t}\left[(A-B)\sin 0.1t + (A+B)\cos 0.1t\right]$

$\Rightarrow (14-6)\sin 0.1t + 20\cos 0.1t = 0$

$\div \cos 0.1t$

$$\frac{8\sin 0.1t}{\cos 0.1t} + 20 = 0$$

$\tan 0.1t = \frac{-5}{2}$

$0.1t = \tan^{-1}\left(\frac{-5}{2}\right) = -1.19$

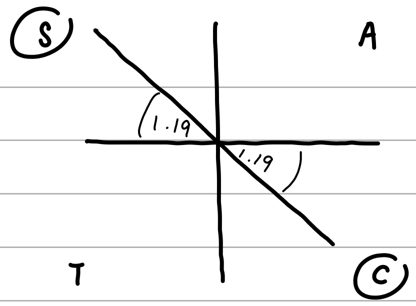
Question 7 continued

$(t>0)$

$\therefore 0.1t = \pi - 1.19$

$t = 10(\pi - 1.19)$

$\simeq 19.5$



19.5 years after 2000 $\rightarrow$ $\boxed{2019}$

ii) at $t = 19.5$, $f = e^{0.3(19.5)}\left[(14\sin(0.1\times 19.5) + 6\cos(0.1\times 19.5)\right]$

$= 3754.5$

$\simeq \boxed{3750}$ to 3 s.f.

iii) model suggests that there will be lots of foxes alive when the rabbits die out which isn't likely to be accurate.

Question 7 continued

(Total for Question 7 is 17 marks)

TOTAL FOR PAPER IS 75 MARKS